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Predicting neutrino parameters from SO(3) family symmetry and quark-lepton unification

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Abstract: We show how the neutrino mixing angles and oscillation phase can be predicted from tri-bimaximal neutrino mixing, corrected by charged lepton mixing angles which are related to quark mixing angles via quark-lepton unification. The tri-bimaximal neutrino mixing can naturally originate from the see-saw mechanism via constrained sequential dominance (CSD), where CSD can result from the vacuum alignment of a non-abelian family symmetry such as SO(3). We construct a realistic model of quark and lepton masses and mixings based on SO(3) family symmetry with quark-lepton unification based on the Pati-Salam gauge group. The atmospheric angle is predicted to be approximately maximal $\theta_{23} = 45°$, corrected by the quark mixing angle $\theta_{23}^{\mathrm{CKM}} \approx 2.4°$, with the correction controlled by an undetermined phase in the quark sector. The solar angle is predicted by the tri-bimaximal complementarity relation: $\theta_{12} + \frac{1}{\sqrt{2}}\frac{\theta_{\mathrm{C}}}{3}\cos(\delta - \pi) \approx 35.26°$, where θ_{C} is the Cabibbo angle and δ is the neutrino oscillation phase. The reactor angle is predicted to be $\theta_{13} \approx \frac{1}{\sqrt{2}}\frac{\theta_{\mathrm{C}}}{3} \approx 3.06°$. The MNS neutrino oscillation phase δ is predicted in terms of the solar angle to be $\cos(\delta - \pi) \approx (35.26° - \theta_{12}^{\circ})/3.06°$. These predictions can all be tested by future high precision neutrino oscillation experiments, thereby probing the nature of high energy quark-lepton unification.

Keywords: Supersymmetric Standard Model, Neutrino Physics, Beyond Standard Model, GUT.

JHEP08(2005)105

Contents

1. Introduction

The discovery of neutrino masses and mixing angles, arguably the greatest advance in physics over the past decade, has provided new clues in the search for a theory of quark and lepton masses and mixings. For example, it is interesting to compare the observed or bounded lepton mixing angles [1]:

$$\theta_{12} = 33.2^\circ \pm 5^\circ\,, \qquad \theta_{23} = 45^\circ \pm 10^\circ\,, \qquad \theta_{13} < 13^\circ\,, \tag{1.1}$$

to the observed quark mixing angles and phase [2]:

$$\theta_{12}^{\rm CKM} = 13.0^\circ \pm 0.1^\circ\,, \quad \theta_{23}^{\rm CKM} = 2.4^\circ \pm 0.1^\circ\,, \quad \theta_{13}^{\rm CKM} = 0.2^\circ \pm 0.1^\circ\,, \quad \delta^{\rm CKM} = 60^\circ \pm 14^\circ\,. \tag{1.2}$$

The quest to understand the relation between the very different lepton and quark mixing angles has led to a great deal of theoretical model building [3]. The poorly determined lepton parameters (especially the neutrino oscillation phase δ which is completely undetermined) as compared to the quark mixing angles, presents an opportunity to make testable predictions in the lepton sector by relating the lepton mixing parameters to the quark ones. This can provide theoretical motivation for making high precision measurements in the neutrino sector. For instance the empirical relation between the leptonic mixing angle θ_{12} (the solar angle) and the Cabibbo angle $\theta_{\rm C} = \theta_{12}^{\rm CKM}$

$$\theta_{12} + \theta_{\rm C} \approx \frac{\pi}{4} \tag{1.3}$$

has recently been the subject of much speculation [4 – 10]. The interest arises from the possibility that this so-called quark-lepton complementarity (QLC) relation could be a signal of some high scale quark-lepton unification. All the attempts to reproduce the QLC relation in the literature so far start from some kind of maximal or bi-maximal mixing in either the neutrino or the charged lepton sectors, then consider the corrections to maximal mixing coming from the other sector.

For example, in the context of inverted hierarchy models with a pseudo-Dirac structure, it was observed some time ago [11, 12] that the predictions for the neutrino mixing angles of $\theta^\nu_{12} = \pi/4$, $\theta^\nu_{13} = 0$, may receive corrections from the charged lepton mixing angle of order the Cabibbo angle, $\theta^e_{12} \sim \theta_{\mathrm{C}}$, resulting in θ_{12} being in the LMA MSW range, and θ_{13} close to its current experimental limit. Recently [9] it was shown that such a scheme, when combined with a Pati-Salam symmetry, could lead to approximate QLC. However the way that this was achieved was quite non-trivial. The contribution to θ_{12} coming from the charged lepton mixing angle θ^e_{12} is suppressed by a factor of $1/\sqrt{2}$ [12], due to the approximately maximal atmospheric mixing angle, and an approximate QLC relation was achieved by selecting operators which give rise to $\theta^e_{12} \approx (3/2)\theta_C$, enhancing the charged lepton mixing angle by a Clebsch factor of 3/2, in order to approximately cancel the suppression factor of $1/\sqrt{2}$ [9]. This approach leads to the predictions $m_\mu/m_s = 2$ at the GUT scale, and the "reactor" leptonic mixing angle $\theta_{13} \approx \theta_C$, both of which are on the edge of current experimental limits. The traditional expectation from unified models that $\theta^e_{12} \approx \theta_{\mathrm{C}}/3$, corresponding to the Georgi-Jarlskog (GJ) [13] relation $m_\mu/m_s = 3$ at the GUT scale, while being more consistent with data, is clearly inconsistent with the above approach to QLC. This motivates the search for alternative models of QLC which would be consistent with the GJ relations.

In this paper we discuss QLC from a completely different starting point, namely tri-bimaximal neutrino mixing [14]. We emphasise that, unlike [14], tri-bimaximal mixing *in the neutrino sector* is merely a staging point in our considerations, and the final form of the lepton mixing matrix, after charged lepton mixing angles have been taken into account, will not have the tri-bimaximal form. We therefore refer to this approach as tri-bimaximal *complementarity* to distinguish it from the usual tri-bimaximal neutrino mixing. To be precise we shall derive from the see-saw mechanism a *neutrino* mixing matrix of the tri-bimaximal form:

$$V^\dagger_{\nu_{\mathrm{L}}} \approx \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{1.4}$$

Then, in the conventions of appendix A, the MNS matrix is given by $U_{\mathrm{MNS}} = V_{e_{\mathrm{L}}} V^\dagger_{\nu_{\mathrm{L}}}$, and so the MNS matrix will not be of the tri-bimaximal form but will involve a left multiplication by the charged lepton mixing matrix $V_{e_{\mathrm{L}}}$. Whereas the neutrino mixing angles arising from tri-bimaximal mixing take the approximate values $\theta^\nu_{12} = \sin^{-1}(1/\sqrt{3}) = 35.26°$, $\theta^\nu_{23} = 45°$, $\theta^\nu_{13} = 0°$, the physical lepton mixing angles arising from tri-bimaximal *complementarity* will differ from these values due to the charged lepton mixing angle corrections, which in turn are related to the quark mixing angles. The atmospheric angle is predicted to be approximately maximal $\theta_{23} = 45°$, corrected by the quark mixing angle $\theta^{\mathrm{CKM}}_{23} \approx 2.4°$, with the correction controlled by an undetermined phase in the quark sector. The reactor angle is predicted to be $\theta_{13} \approx \frac{1}{\sqrt{2}}\frac{\theta_C}{3} \approx 3.06°$.[1] The solar angle is predicted from the tri-bimaximal complementarity relation,

$$\theta_{12} + \frac{1}{\sqrt{2}}\frac{\theta_C}{3}\cos(\delta - \pi) \approx 35.26° \,. \tag{1.5}$$

In eq. (1.5) the factor of $1/3$ arises from the GJ relations, the factor of $1/\sqrt{2}$ arises from the atmospheric angle as discussed previously, and δ is the MNS oscillation phase. The tri-bimaximal complementarity relation in eq. (1.5) may be compared to the bimaximal complementarity relation eq. (1.3). The two relations are approximately numerically equivalent in the case that $\delta = \pi$ since $\frac{1}{\sqrt{2}}\frac{\theta_C}{3} \approx 3.06°$.[2] The tri-bimaximal complementarity relation has the twin advantages that it incorporates the GJ relations, as well as the factor of $1/\sqrt{2}$ which proves troublesome for bi-maximal complementarity. In addition eq. (1.5) may be used to predict the neutrino oscillation phase δ from a future accurate measurement of the solar angle θ_{12}, $\cos(\delta-\pi) \approx (35.26° - \theta_{12}^{\circ})/3.06°$. These predictions can be tested by future high precision neutrino oscillation experiments.

There has recently been some progress with achieving tri-bimaximal neutrino mixing from the see-saw mechanism using vacuum alignment with various family symmetries such as SU(3) [16] or the discrete symmetry A_4 [17]. Here we shall show how tri-bimaximal neutrino mixing can emerge in a natural and general way from the see-saw mechanism using sequential dominance [18], with certain simple constraints imposed on the Yukawa couplings, independently of any particular choice of family symmetry. We refer to this general approach as constrained sequential dominance (CSD). CSD can arise from the vacuum alignment some non-abelian family symmetry, and here we focus on SO(3). A potential advantage of using SO(3) family symmetry is that it is possible to "up-grade" any resulting model of hierarchical neutrino masses to a type-II see-saw model with a quasi-degenerate spectrum of neutrino masses [19], and improved prospects for leptogenesis [20], although in this paper we shall restrict ourselves to hierarchical neutrino masses. We shall subsequently present an explicit model based on SO(3) family symmetry and Pati-Salam unification which is consistent with all quark and lepton masses and mixings, and gives rise to tri-bimaximal *complementarity* using the Georgi-Jarlskog relations. All types of complementarity are crucially dependent on the plethora of complex phases which are generally present in the Yukawa matrices. Here we shall keep careful track of all the phases and in our approach show how tri-bimaximal complementarity may be linked to the MNS CP violating phase.

[1] This prediction for the reactor angle also follows from bi-maximal neutrino mixing, in which $\theta_{13}^{\nu} = 0$, and θ_{13} originates from charged lepton mixing angle of order one third of the Cabibbo angle, $\theta_{12}^{e} \sim \theta_C/3$, typical of the GJ correction [15]. However the solar angle cannot be accounted for by such charged lepton corrections in bi-maximal neutrino mixing [15].

[2] Typically the bi-maximal complementarity relation in eq. (1.3) will also involve a similar phase which is often neglected without good reason. Note that the terminology "tri-bimaximal complementarity" as introduced here is a short-hand for "charged lepton corrections to tri-bimaximal neutrino mixing in which the charged lepton mixing angles are related to the quark mixing angles". In particular it refers to the relation in eq. (1.5)."

This paper has been organized as follows: in section 2, we discuss how to achieve tri-bimaximal mixing using CSD, and briefly show how CSD could arise from vacuum alignment with an SO(3) family symmetry. In section 3 we present an explicit model based on SO(3) famly symmetry and Pati-Salam unification which has all the necessary ingredients that we require. In section 4 we discuss the predictions arising from the model, including a careful discussion of the complex phases which appear in tri-bimaximal complementarity. Section 5 concludes the paper. In appendix A we specify our conventions, while in appendix B we discuss vacuum alignment in the model.

2. Tri-bimaximal neutrino mixing from the see-saw mechanism with constrained sequential dominance

The fact that the tri-bimaximal neutrino mixing matrix in eq. (1.4) involves square roots of simple ratios motivates models in which the mixing angles are independent of the mass eigenvalues. One such class of models are see-saw models with sequential dominance (SD) of right-handed neutrinos [12, 18]. In SD, the atmospheric and solar neutrino mixing angles are determined in terms of ratios of Yukawa couplings involving the dominant and subdominant right-handed neutrinos, respectively. If these Yukawa couplings are simply related in some way, then it is possible for simple neutrino mixing angle relations, such as appear in tri-bimaximal neutrino mixing, to emerge in a simple and natural way, independently of the neutrino mass eigenvalues.

To see how tri-bimaximal neutrino mixing could emerge from SD, we begin by writing the right-handed neutrino Majorana mass matrix $M_{\rm RR}$ in a diagonal basis as

$$M_{\rm RR} \approx \begin{pmatrix} Y & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & X' \end{pmatrix}, \tag{2.1}$$

where we shall assume

$$Y \ll X \ll X' \,. \tag{2.2}$$

Then in this basis we write the neutrino (Dirac) Yukawa matrix Y^{ν}_{LR} in terms of the complex Yukawa couplings $a, b, c, d, e, f, a', b', c'$ as

$$Y^{\nu}_{\rm LR} = \begin{pmatrix} d & a & a' \\ e & b & b' \\ f & c & c' \end{pmatrix}, \tag{2.3}$$

in the convention where the Yukawa matrix corresponds to the lagrangian coupling $\bar{L}H_u Y^{\nu}_{LR}\nu_R$, where L are the left-handed lepton doublets, H_u is the Higgs doublet coupling to up-type quarks and neutrinos, and ν_R are the right-handed neutrinos. The Dirac neutrino mass matrix is then given by $m^{\nu}_{\rm LR} = Y^{\nu}_{LR} v_{\rm u}$, where $v_{\rm u}$ is the vacuum expectation value (VEV) of H_u.

JHEP08(2005)105

For simplicity we shall henceforth assume that $d = 0$, although this is not strictly necessary [18]. Then the condition for sequential dominance (SD) is that the right-handed neutrino of mass Y gives the dominant contribution to the see-saw mechanism, while the right-handed neutrino of mass X gives the leading sub-dominant contribution [18]

$$\frac{|e^2|, |f^2|, |ef|}{Y} \gg \frac{|xy|}{X} \gg \frac{|x'y'|}{X'} \tag{2.4}$$

where $x, y \in a, b, c$ and $x', y' \in a', b', c'$, and all Yukawa couplings are assumed to be complex. The combination of eqs. (2.2), (2.4) is called light sequential dominance (LSD) since the lightest right-handed neutrino makes the dominant contribution to the see-saw mechanism. LSD is motivated by unified models in which only small mixing angles are present in the Yukawa sector, and implies that the heaviest right-handed neutrino of mass X' is irrelevant for both leptogenesis and neutrino oscillations (for a discussion of all these points see [3]). In addition many realistic models in the literature (see for example [22]) involve an approximate texture zero in the 11 position, corresponding to our simplifying assumption $d = 0$. This will have the effect of removing one of the see-saw phases.

Assuming eq. (2.4) the neutrino masses are given to leading order in m_2/m_3 by the results in [12], summarized as:

$$m_1 \sim O(\frac{x'y'}{X'})v_u^2 \tag{2.5}$$

$$m_2 \approx \frac{|a|^2}{X(s_{12}^\nu)^2}v_u^2 \tag{2.6}$$

$$m_3 \approx \frac{(|e|^2 + |f|^2)}{Y}v_u^2 \tag{2.7}$$

where $s_{12}^\nu = \sin\theta_{12}^\nu$ may be obtained from the further results given below. Note that with SD each neutrino mass is generated by a separate right-handed neutrino, and the sequential dominance condition naturally results in a neutrino mass hierarchy $m_1 \ll m_2 \ll m_3$. The neutrino mixing angles are given to leading order as [12],

$$\tan\theta_{23}^\nu \approx \frac{|e|}{|f|} \tag{2.8}$$

$$\tan\theta_{12}^\nu \approx \frac{|a|}{c_{23}^\nu|b|\cos(\phi_b') - s_{23}^\nu|c|\cos(\phi_c')} \tag{2.9}$$

$$\theta_{13}^\nu \approx e^{i(\phi_2^\nu + \phi_a - \phi_e)}\frac{|a|(e^*b + f^*c)}{[|e|^2 + |f|^2]^{3/2}}\frac{Y}{X} \tag{2.10}$$

where we have written some (but not all) complex Yukawa couplings as $x = |x|e^{i\phi_x}$. The phase χ^ν is fixed to give a real angle θ_{12}^ν by,

$$c_{23}^\nu|b|\sin(\phi'_b) \approx s_{23}^\nu|c|\sin(\phi'_c) \tag{2.11}$$

where

$$\begin{aligned} \phi_b' &\equiv \phi_b - \phi_a - \phi_2^\nu - \chi^\nu\,, \\ \phi_c' &\equiv \phi_c - \phi_a + \phi_e - \phi_f - \phi_2^\nu - \chi^\nu\,. \end{aligned} \tag{2.12}$$

The phase ϕ_2^ν is fixed to give a real angle θ_{13}^ν by [12],

$$\phi_2^\nu \approx \phi_e - \phi_a - \phi_{\mathrm{COSMO}} \tag{2.13}$$

where

$$\phi_{\mathrm{COSMO}} = \arg(e^* b + f^* c) \tag{2.14}$$

is the leptogenesis phase corresponding to the interference diagram involving the lightest and next-to-lightest right-handed neutrinos [12]. The auxiliary phases appearing above are defined in appendix A.

We can now ask what are the conditions for achieving tri-bimaximal neutrino mixing as in eq. (1.4), in which $\tan\theta_{23}^\nu = 1$, $\tan\theta_{12}^\nu = 1/\sqrt{2}$ and $\theta_{13}^\nu = 0$ in the framework of sequential dominance? Note that in sequential dominance the mixing angles are determined by ratios of Yukawa couplings, and are independent of the neutrino masses. We propose the following set of conditions which are sufficient to achieve tri-bimaximal mixing within the framework of sequential dominance:

$$|a| = |b| = |c|\,, \tag{2.15a}$$
$$|d| = 0\,, \tag{2.15b}$$
$$|e| = |f|\,, \tag{2.15c}$$
$$\phi'_b = 0\,, \tag{2.15d}$$
$$\phi'_c = \pi\,. \tag{2.15e}$$

Eqs. (2.15a), (2.15b), (2.15c) are conditions on the magnitudes of the Yukawa couplings, while eqs. (2.15d), (2.15e) are generic phase conditions which can be satisfied by several different types of phase structure in the Yukawa matrix. The condition in eq. (2.15c) clearly gives rise to $\tan\theta_{23}^\nu = 1$, as can be seen from eq. (2.8). The remaining conditions in eq. (2.15) result in $\tan\theta_{12}^\nu = 1/\sqrt{2}$ as can be seen from eq. (2.9). Eqs. (2.15d) and (2.15e), together with the definitions in eq. (2.12), imply the condition on the phases of the Yukawa couplings:

$$\phi_c - \phi_b + \phi_e - \phi_f = \pi\,. \tag{2.16}$$

Eq. (2.16), together with eqs. (2.15c), (2.15a), then implies that:

$$e^* b + f^* c = 0\,. \tag{2.17}$$

Eq. (2.17) implies from eq. (2.10) that $\theta_{13}^\nu = 0$. It also implies that leptogenesis is zero at leading order, independently of the choice of charged lepton basis [12]. We conclude that the conditions in eq. (2.15), together with the conditions for sequential dominance, are sufficient to result in tri-bimaximal neutrino mixing as in eq. (1.4). We shall refer to this as constrained sequential dominance (CSD). Note that, with the conditions in eq. (2.15) satisfied, the angle θ_{12}^ν is automaticaly real, so the phase χ^ν is undetermined, and will be expected to play no part in physics. The phase ϕ_2^ν is similarly undetermined and unphysical, since $\theta_{13}^\nu = 0$.

JHEP08(2005)105

Since there are undetermined phases above, it is instructive to consider tri-bimaximal mixing as a limiting case of a known example where the phases are determined. The example will also serve as an introduction to the model of quark and lepton masses and mixings discussed in the next section based on SO(3) family symmetry and Pati-Salam unification, in which a neutrino Yukawa matrix which satisfies the conditions of CSD, will arise. It should be emphasised that other examples based on SU(3) or discrete family symmetries may also give rise to CSD, the general conditions for which are given in eqs. (2.15), (2.16).

Consider a supersymmetric theory in which the lepton doublets L are triplets of an SO(3) family symmetry, but the (CP conjugates of) right-handed neutrinos ν_i^c and Higgs doublets H_u are singlets under the family symmetry [19]. Yukawa couplings arise from the superpotential terms of the form:

$$|y_1|e^{i\delta_1}LH_u\nu_1^c\frac{\phi_{23}}{M}+|y_2|e^{i\delta_2}LH_u\nu_2^c\frac{\phi_{123}}{M}+|y_3|e^{i\delta_3}LH_u\nu_3^c\frac{\phi_3}{M} \tag{2.18}$$

where $\phi_{23}, \phi_{123}, \phi_3$ are SO(3) triplet flavon fields whose vacuum expectation values (VEVs) break the SO(3) family symmetry, and allow Dirac neutrino mass terms to be generated. We have written the Yukawa couplings in terms of magnitudes and phases $|y_i|$ and $e^{i\delta_i}$, and M is a real positive mass scale. Each term in eq. (2.18) only involves a particular flavon superfield coupling together with a particular right-handed neutrino superfield. This may readily be enforced by symmetries, as we shall discuss later in the framework of the Pati-Salam theory.

In [19, 21] it was also shown how to generate real flavon VEVs:

$$\tfrac{|y_2|\phi_{123}}{M}=\begin{pmatrix}a\\b\\c\end{pmatrix},\qquad \tfrac{|y_1|\phi_{23}}{M}=\begin{pmatrix}0\\e\\f\end{pmatrix},\qquad \tfrac{|y_3|\phi_3}{M}=\begin{pmatrix}0\\0\\c'\end{pmatrix}. \tag{2.19}$$

When these VEVs are inserted into the couplings in eq. (2.18) this results in a neutrino Yukawa matrix:

$$Y^{\nu}_{\rm LR}=\begin{pmatrix}0 & ae^{i\delta_2} & 0\\ ee^{i\delta_1} & be^{i\delta_2} & 0\\ fe^{i\delta_1} & ce^{i\delta_2} & c'e^{i\delta_3}\end{pmatrix}, \tag{2.20}$$

where here a, b, c, e, f, c' are real (positive or negative) numbers. In order to satisfy the CSD constraints in eqs. (2.15), (2.16) it is sufficient to show that it is possible to arrange for the real VEVs in eq. (2.19) to be aligned such that:

$$e=-f\,, \tag{2.21a}$$
$$a=b=c\,. \tag{2.21b}$$

The phases required to ensure positive neutrino mixing angles are then given in the limit of such a vacuum alignment by:[3]

$$\phi_b'=-\phi_2^{\nu}-\chi^{\nu}=0\,,$$

[3]The undetermined phases ϕ_2^{ν},χ^{ν} are specified above by slightly relaxing the conditions in eqs. (2.21a), (2.21b), while keeping $|b|>|c|$. However these phases are unphysical in the case of tri-bimaximal neutrino mixing and they could equally well be set to zero.

JHEP08(2005)105

$$\phi_c' = \pi - \phi_2^\nu - \chi^\nu = \pi\,, \tag{2.22a}$$
$$\phi_2^\nu = 2(\delta_1 - \delta_2)\,, \tag{2.22b}$$
$$\phi_3^\nu = \phi_2^\nu + \pi\,, \tag{2.22c}$$
$$\omega_1^\nu = \delta_3\,, \tag{2.22d}$$
$$\omega_2^\nu = \delta_2\,, \tag{2.22e}$$
$$\omega_3^\nu = \delta_1\,, \tag{2.22f}$$

where the phases are defined in appendix A. Such a vacuum alignment would then satisfy the constraints in eqs. (2.15), (2.16). In order to arrange for the VEVs in eqs. (2.21a), (2.21b), we need to introduce additional flavon superfields and superpotential terms involving these and other superfields, as discussed in appendix B. It is clear that SO(3) family symmetry and vacuum alignment can provide a realization of CSD and hence tri-bimaximal *neutrino* mixing. To obtain tri-bimaximal *complementarity* we also require quark-lepton unification which incorporates the GJ relations, and we now turn to the construction of such a model.

3. SO(3) family symmetry and Pati-Salam unification

We now construct a realistic model based on a family symmetry SO(3) and Pati-Salam unification. The model will incorporate both the vacuum alignment in SO(3) necessary to achieve tri-bimaximal neutrino mixing via constrained sequential dominance, and also will involve the GJ relations necessary to relate the charged lepton mixing angle to the Cabibbo angle. The explicit model will demonstrate that all these features can be achieved together within a single consistent framework, and will lead to further relations between the lepton and quark mixing angles.

The model is a supersymmetric theory based on the family symmetry SO(3) together with the Pati-Salam gauge group [23]

$$G_{PS} = \mathrm{SU}(4)_{\mathrm{PS}} \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}}\,. \tag{3.1}$$

Assuming the Pati-Salam symmetry to start with has the advantage that it explicitly exhibits $\mathrm{SU}(4)_{PS}$ quark-lepton and $\mathrm{SU}(2)_R$ isospin symmetry, allowing Georgi-Jarlskog factors to be generated and isospin breaking to be controlled, while avoiding the Higgs doublet-triplet splitting problem [24]. Quarks and leptons are unified in the $\mathrm{SU}(4)_{\mathrm{PS}}$-quartets F_i and F_i^c of $\mathrm{SU}(4)_{\mathrm{C}}$, which are doublets of $\mathrm{SU}(2)_{\mathrm{L}}$ and $\mathrm{SU}(2)_{\mathrm{R}}$, respectively,

$$F_i = \begin{pmatrix} u_i & u_i & u_i & \nu_i \\ d_i & d_i & d_i & e_i \end{pmatrix}, \qquad F_j^c = \begin{pmatrix} u_j^c & u_j^c & u_j^c & \nu_j^c \\ d_j^c & d_j^c & d_j^c & e_j^c \end{pmatrix}, \tag{3.2}$$

where i and j are family indices. In addition the left-handed quarks and leptons are assigned to be triplets, while the CP-conjugates of the right-handed quarks and leptons are singlets under an SO(3) family symmetry,

$$F_i \sim \mathbf{3}\,, \qquad F_j^c \sim \mathbf{1}\,. \tag{3.3}$$

Field	SO(3)	SU(4)$_{PS}$	SU(2)$_L$	SU(2)$_R$	R	U(1)	Z_4^I	Z_4^{II}	Z_3^I	Z_3^{II}
F_i	3	4	2	1	1	0	0	0	0	0
F_1^c	1	$\overline{4}$	1	2	1	-3	α	0	0	0
F_2^c	1	$\overline{4}$	1	2	1	-3	0	β	0	0
F_3^c	1	$\overline{4}$	1	2	1	0	0	0	0	0
h	1	1	2	2	0	0	0	0	0	0
H	1	4	1	2	0	0	0	0	0	0
$\overline{H}$	1	$\overline{4}$	1	2	0	0	0	0	0	0
Σ	1	15	1	3	0	2	α^3	β^3	0	0
ϕ_1	3	1	1	1	0	0	0	0	γ	0
ϕ_2	3	1	1	1	0	0	0	0	0	δ
ϕ_3	3	1	1	$3 \oplus 1$	0	0	0	0	0	0
ϕ_{23}	3	1	1	1	0	1	α	0	0	0
ϕ_{123}	3	1	1	1	0	1	0	β	0	0

Table 1: Transformation of the superfields under the SO(3) family, Pati-Salam and $R \times Z_4^2 \times Z_3^2 \times$ U(1) symmetries which restrict the form of the mass matrices. The continuous R-symmetry may be alternatively be replaced by a discrete Z_{2R} symmetry. We only display the fields relevant for generating fermion mass and spontaneous symmetry breaking.

This implies in particular that the right-handed neutrinos $\nu_j^c \in F_j^c$ are singlets under SO(3), and the lepton electroweak doublets $L_i \in F_i$ are triplets under SO(3), as assumed previously. The usual SUSY Higgs doublets H_u, H_d are contained in a single PS bi-doublet h, and further heavy Higgs superfields $H, \overline{H}$ are introduced to break the Pati-Salam symmetry group down to the Standard Model [25]. As in the SU(3) model in [16], we include an adjoint Σ field which develops vevs in the SU(4)$_{PS} \times$ SU(2)$_R$ direction which preserves the hypercharge generator $Y = T_{3R} + (B-L)/2$. This implies that any coupling of the Σ to a fermion and a messenger such as $\Sigma_{b\beta}^{a\alpha} F_{a\alpha}^c \chi^{b\beta}$, where the SU(2)$_R$ and SU(4)$_{PS}$ indices have been displayed explicitly, is proportional to the hypercharge Y of the particular fermion component of F^c times the vev σ. For example the coupling of Σ to right-handed neutrinos gives zero. In addition to SO(3) $\times G_{PS}$, the flavour symmetry group also includes $R \times Z_4^2 \times Z_3^2 \times$ U(1) symmetries in order to restrict the form of the mass matrices, where the continuous R-symmetry may be alternatively be replaced by a discrete Z_{2R} symmetry. The superfields transform under the full symmetry group of the model as shown in table 1.

We need spontaneous breaking of the family symmetry

$$\mathrm{SO}(3) \longrightarrow \mathrm{SO}(2) \longrightarrow \mathrm{Nothing}\,. \tag{3.4}$$

To achieve this symmetry breaking we introduce additional flavon fields ϕ_i, ϕ_{23}, ϕ_{123} in the representations given in table 1. The vacuum alignment of the flavon VEVs plays a crucial role in this model. In the SO(3) model the flavon VEVs are all real and, as discussed in appendix B, may be aligned in the following way:

$$\phi_1 \sim \begin{pmatrix}1\\0\\0\end{pmatrix}, \qquad \phi_2 \sim \begin{pmatrix}0\\1\\0\end{pmatrix}, \qquad \phi_3 \sim \begin{pmatrix}0\\0\\1\end{pmatrix}, \qquad \phi_{23} \sim \begin{pmatrix}0\\1\\-1\end{pmatrix}, \qquad \phi_{123} \sim \begin{pmatrix}1\\1\\1\end{pmatrix}. \tag{3.5}$$

The leading operators allowed by the symmetries are

$$\begin{aligned}
W_{\rm Yuk} &= \frac{y_1}{M^3}(F.\phi_{23})\phi_{23}^2 F_1^c h + && (3.6)\\
&+\frac{y_2}{M^3}(F.\phi_{123})\phi_{123}^2 F_2^c h + \frac{y_2'\Sigma}{M^2}(F.\phi_{23})F_2^c h + && (3.7)\\
&+\frac{y_3}{M}(F.\phi_3)F_3^c h + \frac{y_3'}{M^3}(F.\phi_2)\phi_2^2 F_3^c h + \frac{y_3''}{M^3}(F.\phi_1)\phi_1^2 F_3^c h && (3.8)\\
W_{\rm Maj} &\sim \frac{1}{M}(F_3^c H)^2 + && (3.9)\\
&+\frac{1}{M^7}(F_2^c H)^2(\phi_{123}^2\phi_{23}^4 + \phi_{123}^6) + && (3.10)\\
&+\frac{1}{M^7}(F_1^c H)^2(\phi_{23}^6 + \phi_{23}^2\phi_{123}^4) + && (3.11)\\
&+\frac{1}{M^5}(F_2^c H)(F_3^c H)\phi_{123}^3\phi_3 + && (3.12)\\
&+\frac{1}{M^5}(F_1^c H)(F_3^c H)\phi_{23}^3\phi_3 + && (3.13)\\
&+\frac{1}{M^7}(F_1^c H)(F_2^c H)\phi_{23}^3\phi_{123}^3\,, && (3.14)
\end{aligned}$$

where we have included complex Yukawa couplings $y_i = |y_i|e^{i\delta_i}$ in the Yukawa superpotential, but have suppressed similar Yukawa couplings which would appear multiplying the Majorana operators.

In order to obtain the Yukawa matrices from $W_{\rm Yuk}$ and the Majorana matrix from $W_{\rm Maj}$ requires some discussion of the messenger sector that is responsible for the operators above. This was fully discussed in [16], and we shall only briefly repeat the essential points here. The operators arise from Froggat-Nielsen diagrams and the scale M represents the right-handed up and down messenger mass scales $M^{u,d}$, corresponding to the dominance of right-handed messengers over left-handed messengers, which applies if $M < M^L$ where M^L represents the left-handed messenger mass scale. Specifically it was assumed that

$$M^d \approx \frac{1}{3}M^u \ll M^L\,. \tag{3.15}$$

It was further assumed that the right-handed lepton messenger scales satisfy the approximate $SU(4)_{PS}$ relations $M^\nu \simeq M^u$, and $M^e \simeq M^d$. The splitting of the messenger mass scales relies on left-right and $SU(2)_R$ breaking effects which was assumed to be due to a Wilson line symmetry breaking mechanism [16]. Eq. (3.15) then allows the expansion parameters associated with ϕ_{23} to take the numerical values [16]:

$$\epsilon \equiv \frac{\phi_{23}}{M^u} \approx 0.05\,, \qquad \bar\epsilon \equiv \frac{\phi_{23}}{M^d} \approx 0.15 \tag{3.16}$$

where here and henceforth we assume that the fields have been replaced by their VEVs e.g. $\phi_{23} \to \langle\phi_{23}\rangle$, etc. We shall also assume that the flavons ϕ_{123} take similar VEVs:

$$\frac{\phi_{123}}{M^u} \approx \epsilon\,, \qquad \frac{\phi_{123}}{M^d} \approx \bar{\epsilon} \tag{3.17}$$

In the present model the flavons $\phi_{1,2}$ lead to independent expansion parameters which we will assume to satisfy:

$$\frac{\phi_1}{M^d} \approx \bar{\epsilon}\,, \qquad \frac{\phi_2}{M^d} \approx \bar{\epsilon}^{2/3}\,. \tag{3.18}$$

Eqs. (3.16), (3.18) then imply:

$$\frac{\phi_1}{M^u} \approx \epsilon\,, \qquad \frac{\phi_2}{M^u} \approx \frac{\bar{\epsilon}^{2/3}}{3} \approx 0.7\epsilon^{2/3}\,. \tag{3.19}$$

The flavon ϕ_3 transforms under $\mathrm{SU}(2)_R$ as $\mathbf{3} \oplus \mathbf{1}$, and develops isospin breaking vevs in the up and down $\mathrm{SU}(2)_R$ directions, and we assume as in [16]:

$$\frac{\phi_3^u}{M^u} = \frac{\phi_3^d}{M^d} \approx \sqrt{\bar{\epsilon}}\,. \tag{3.20}$$

It remains to specify the expansion parameter associated with σ, the vev of Σ. This was determined purely by phenomenological considerations in [16], and here we assume the same value:

$$Y(d)\frac{\sigma}{M^d} \approx \bar{\epsilon}\,. \tag{3.21}$$

Note that the operators involving Σ must be multiplied by the hypercharge of the relevant right-handed fermion, where $Y(d) = 1/3$ is the hypercharge of d^c, $Y(u) = -2/3$ is the hypercharge of u^c, $Y(e) = 1$ is the hypercharge of e^c, and $Y(\nu) = 0$ is the hypercharge of ν^c.

The operators in eqs. (3.6), (3.7), (3.8) when combined with the messenger sector just described, then leads to the Yukawa matrices:

$$Y^U_{LR} \approx \begin{pmatrix} 0 & y_2\epsilon^3 & y_3''\epsilon^3 \\ y_1\epsilon^3 & y_2\epsilon^3 - 2y_2'\epsilon^2 & 0.34y_3'\epsilon^2 \\ -y_1\epsilon^3 & y_2\epsilon^3 + 2y_2'\epsilon^2 & y_3\bar{\epsilon}^{\frac{1}{2}} \end{pmatrix}, \tag{3.22}$$

$$Y^D_{LR} \approx \begin{pmatrix} 0 & y_2\bar{\epsilon}^3 & y_3''\bar{\epsilon}^3 \\ y_1\bar{\epsilon}^3 & y_2\bar{\epsilon}^3 + y_2'\bar{\epsilon}^2 & y_3'\bar{\epsilon}^2 \\ -y_1\bar{\epsilon}^3 & y_2\bar{\epsilon}^3 - y_2'\bar{\epsilon}^2 & y_3\bar{\epsilon}^{\frac{1}{2}} \end{pmatrix}, \tag{3.23}$$

$$Y^E_{LR} \approx \begin{pmatrix} 0 & y_2\bar{\epsilon}^3 & y_3''\bar{\epsilon}^3 \\ y_1\bar{\epsilon}^3 & y_2\bar{\epsilon}^3 + 3y_2'\bar{\epsilon}^2 & y_3'\bar{\epsilon}^2 \\ -y_1\bar{\epsilon}^3 & y_2\bar{\epsilon}^3 - 3y_2'\bar{\epsilon}^2 & y_3\bar{\epsilon}^{\frac{1}{2}} \end{pmatrix}, \tag{3.24}$$

$$Y^\nu_{LR} \approx \begin{pmatrix} 0 & y_2\epsilon^3 & y_3''\epsilon^3 \\ y_1\epsilon^3 & y_2\epsilon^3 & 0.34y_3'\epsilon^2 \\ -y_1\epsilon^3 & y_2\epsilon^3 & y_3\bar{\epsilon}^{\frac{1}{2}} \end{pmatrix}. \tag{3.25}$$

The leading corrections are given by additional operators similar to those displayed but with insertions of powers of $\phi_3^2/M^2 \sim \bar{\epsilon}$.

The leading heavy right-handed neutrino Majorana mass arises from the operator of eq. (3.9) which gives,

$$M_3 \approx \frac{< H >^2}{M^\nu}\,, \tag{3.26}$$

to the third family, where $M^\nu = M^u$ is the same messenger mass scale as in the up sector due to $SU(4)_{PS}$. Operators involving Σ do not contribute since it does not couple to right-handed neutrinos which have zero hypercharge. However the other Majorana operators fill out the Majorana mass matrix, and after small angle right-handed rotations the Majorana matrix takes the form:

$$M_{RR} = \begin{pmatrix} p\epsilon^6 & 0 & 0 \\ 0 & q\epsilon^6 & 0 \\ 0 & 0 & 1 \end{pmatrix} M_3\,, \tag{3.27}$$

where p, q are complex couplings.

4. Predictions for Neutrino Parameters

The neutrino Yukawa matrix in eq. (3.25) has the CSD form considered in eqs. (2.20), (2.21) and, provided the SD conditions in eq. (2.4) are satisfied, it will lead to tri-bimaximal neutrino mixing. The complex phases in M_{RR} in eq. (3.27) may be removed by rotations on the right-handed neutrino fields, which only results in a redefinition of the Yukawa phases appearing in the complex Yukawa couplings $y_i = |y_i|e^{i\delta_i}$. Effectively, then, the Majorana masses in M_{RR} may be taken to be real without loss of generality. The SD conditions in eq. (2.4) are then satisfied providing:

$$\frac{|y_1^2|}{p} \gg \frac{|y_2^2|}{q} \gg |y_3^2|\bar{\epsilon}\,. \tag{4.1}$$

Since the Yukawa couplings are expected to be of order unity, the model predicts a rather mild hierarchy in physical neutrino masses, from eqs. (2.5), (2.6), (2.7):

$$m_1 \approx \frac{|y_3^2|\bar{\epsilon}}{M_3} v_u^2 \tag{4.2}$$

$$m_2 \approx \frac{3|y_2^2|}{qM_3} v_u^2 \tag{4.3}$$

$$m_3 \approx \frac{2|y_1^2|}{pM_3} v_u^2\,. \tag{4.4}$$

The neutrino mixing angles take the tri-bimaximal values:

$$\tan\theta_{23}^\nu \approx 1 \tag{4.5}$$

$$\tan\theta_{12}^\nu \approx \frac{1}{\sqrt{2}} \tag{4.6}$$

$$\theta_{13}^\nu \approx 0 \tag{4.7}$$

since the Yukawa matrix in eq. (3.25) has the CSD form considered in eqs. (2.20), (2.21), as already discussed. The neutrino auxiliary phases then take the values given in eq. (2.22), where the phases δ_i refer to the phases of the Yukawa couplings $y_i = |y_i|e^{i\delta_i}$ (assuming without loss of generality real p, q). From eq. (2.22) and eqs. (A.17)–(A.19) we obtain the neutrino phases:

$$\delta_{12}^{\nu_L} = (\delta_3 - \delta_2) \tag{4.8}$$
$$\delta_{13}^{\nu_L} = (\delta_3 - \delta_1) \tag{4.9}$$
$$\delta_{23}^{\nu_L} = -3(\delta_1 - \delta_2)\,. \tag{4.10}$$

The lepton mixing angles receive corrections from the charged lepton sector which in this model are completely derivable from the charged lepton Yukawa matrix in eq. (3.24), using the results in appendix A. The charged lepton Yukawa matrix in eq. (3.24) leads to charged lepton masses in the ratios:

$$m_e : m_\mu : m_\tau \approx \frac{|y_1||y_2|}{3|y_2'|}\bar{\epsilon}^4 : 3|y_2'|\bar{\epsilon}^2 : y_{33}\,. \tag{4.11}$$

In a small charged lepton angle approximation,

$$\theta_{23}^{E_L} \approx \frac{|y_3'|\bar{\epsilon}^2}{y_{33}} \tag{4.12}$$
$$\theta_{13}^{E_L} \approx \frac{|y_3''|\bar{\epsilon}^3}{y_{33}} \tag{4.13}$$
$$\theta_{12}^{E_L} \approx \frac{|y_2|\bar{\epsilon}}{3|y_2'|} \tag{4.14}$$

where we have written $y_{33} = |y_3|\bar{\epsilon}^{\frac{1}{2}}$. The auxiliary charged lepton phases, used to make the charged lepton mixing angles real and positive, are:

$$\phi_2^{E_L} = \delta_3' - \delta_3''\,, \tag{4.15a}$$
$$\phi_3^{E_L} = \delta_3 - \delta_3''\,, \tag{4.15b}$$
$$\chi^{E_L} = \delta_2' - \delta_2 - \delta_3' + \delta_3'' \tag{4.15c}$$

$\omega_i^{E_L}$ are undetermined and are used to remove phases from the MNS matrix. From eq. (2.22), (4.15) and eqs. (A.20)–(A.22) we obtain the charged lepton phases:

$$\delta_{23}^{E_L} = (\delta_3 - \delta_3') - \pi - 3(\delta_1 - \delta_2) \tag{4.16}$$
$$\delta_{13}^{E_L} = (\delta_3 - \delta_3'') + (\delta_3 - \delta_1) - \pi - 2(\delta_1 - \delta_2) \tag{4.17}$$
$$\delta_{12}^{E_L} = (\delta_2' - \delta_2) + (\delta_3 - \delta_2)\,. \tag{4.18}$$

The leading charged lepton corrections to the MNS angles and phases are given from eqs. (A.14)–(A.16):

$$s_{23}e^{-i\delta_{23}} \approx e^{-i\delta_{23}^{\nu_L}}\left[s_{23}^{\nu_L} - \theta_{23}^{E_L}c_{23}^{\nu_L}e^{-i(\delta_{23}^{E_L} - \delta_{23}^{\nu_L})}\right] \tag{4.19}$$
$$\theta_{13}e^{-i\delta_{13}} \approx -\theta_{12}^{E_L}s_{23}^{\nu_L}e^{-i(\delta_{23}^{\nu_L} + \delta_{12}^{E_L})} \tag{4.20}$$
$$s_{12}e^{-i\delta_{12}} \approx e^{-i\delta_{12}^{\nu_L}}\left[s_{12}^{\nu_L} - \theta_{12}^{E_L}c_{23}^{\nu_L}c_{12}^{\nu_L}e^{-i(\delta_{12}^{E_L} - \delta_{12}^{\nu_L})}\right] \tag{4.21}$$

where we have kept the leading charged lepton correction in each term. From eqs. (4.19)–(4.21), we see that the lepton phases are approximately given by:

$$\delta_{23} \approx \delta_{23}^{\nu_L} \tag{4.22}$$

$$\delta_{13} \approx \delta_{23}^{\nu_L} + \delta_{12}^{E_L} + \pi \tag{4.23}$$

$$\delta_{12} \approx \delta_{12}^{\nu_L} \tag{4.24}$$

and hence δ, the MNS CP phase relevant for neutrino oscillations, is given by

$$\delta = \delta_{13} - \delta_{23} - \delta_{12} \approx \delta_{12}^{E_L} - \delta_{12}^{\nu_L} + \pi \approx \delta_2' - \delta_2 + \pi\,. \tag{4.25}$$

It is remarkable to observe that the phase appearing in the leading charged lepton correction to the solar angle in eq. (4.21) is just equal to $\delta - \pi$, where δ is the MNS phase. From eqs. (4.19)–(4.21) and the phases in eqs. (4.8)–(4.10) and eqs. (4.16)–(4.18) and the tri-bimaximal neutrino mixing angles in eqs. (4.5)–(4.7), the lepton mixing angles are given by:

$$s_{23} \approx \frac{1}{\sqrt{2}}\left(1 + \theta_{23}^{E_L}\cos(\delta_3 - \delta_3')\right) \tag{4.26}$$

$$\theta_{13} \approx \frac{\theta_{12}^{E_L}}{\sqrt{2}} \tag{4.27}$$

$$s_{12} \approx \frac{1}{\sqrt{3}}\left(1 - \theta_{12}^{E_L}\cos(\delta - \pi)\right). \tag{4.28}$$

We now turn to the quark sector. The quark Yukawa matrices in eqs. (3.22), (3.23) lead to quark masses in the ratios:

$$m_d : m_s : m_b \approx \frac{|y_1||y_2|}{|y_2'|}\bar{\epsilon}^4 : |y_2'|\bar{\epsilon}^2 : y_{33} \tag{4.29}$$

$$m_u : m_c : m_t \approx \frac{|y_1||y_2|}{2|y_2'|}\epsilon^4 : 2|y_2'|\epsilon^2 : y_{33}\,. \tag{4.30}$$

Comparing the down masses in eq. (4.29) to the charged lepton masses in eq. (4.11) we see the expected GJ relations (valid at the GUT scale):

$$\frac{m_e}{m_d} = \frac{1}{3}\,, \qquad \frac{m_\mu}{m_s} = 3\,, \qquad \frac{m_\tau}{m_b} = 1\,. \tag{4.31}$$

Using the conventions in appendix A, the quark Yukawa matrices in eqs. (3.22), (3.23) lead to mixing angles:

$$\theta_{23}^{U_L} \approx \frac{0.34|y_3'|\epsilon^2}{y_{33}} \tag{4.32}$$

$$\theta_{13}^{U_L} \approx \frac{|y_3''|\epsilon^3}{y_{33}} \tag{4.33}$$

$$\theta_{12}^{U_L} \approx \frac{|y_2|\epsilon}{2|y_2'|} \tag{4.34}$$

$$\theta_{23}^{D_L} \approx \frac{|y_3'|\bar{\epsilon}^2}{y_{33}} \tag{4.35}$$

$$\theta_{13}^{D_L} \approx \frac{|y_3''|\bar{\epsilon}^3}{y_{33}} \tag{4.36}$$

$$\theta_{12}^{D_L} \approx \frac{|y_2|\bar{\epsilon}}{|y_2'|} \tag{4.37}$$

where we have written $y_{33} = |y_3|\bar{\epsilon}^{\frac{1}{2}}$, and we have used a small quark angle approximation. The auxiliary quark phases, used to make the quark mixing angles real and positive, are exactly the same as the charged lepton phases in eq. (4.15), except that χ^{U_L} has an additional phase of π resulting from the negative 22 element of the up quark Yukawa matrix. Using the results in appendix A we find the CKM angles and phase:

$$\theta_{23}^{\rm CKM} \approx \theta_{23}^{D_L} - \theta_{23}^{U_L} \tag{4.38}$$

$$\theta_{13}^{\rm CKM} \approx \theta_{13}^{D_L} \tag{4.39}$$

$$\theta_{12}^{\rm CKM} \approx \theta_{12}^{D_L} + \theta_{12}^{U_L} \tag{4.40}$$

$$\delta^{\rm CKM} \approx (\delta_2' - \delta_2) + (\delta_3' - \delta_3'')\,. \tag{4.41}$$

The CKM phase is equal to the MNS phase in eq. (4.25) plus a second independent phase determined by elements in the third column of the Yukawa matrix which are irrelevant for neutrino mixing.

Since the CKM angles are approximately given by the down quark mixing angles, using eqs. (4.38)–(4.40), eqs. (4.35)–(4.37), eqs. (4.12)–(4.14), we may relate the charged lepton mixing angles to the CKM angles,

$$\theta_{23}^{E_L} \approx \theta_{23}^{D_L} \approx \theta_{23}^{\rm CKM} \tag{4.42}$$

$$\theta_{13}^{E_L} \approx \theta_{13}^{D_L} \approx \theta_{13}^{\rm CKM} \tag{4.43}$$

$$\theta_{12}^{E_L} \approx \frac{1}{3}\theta_{12}^{D_L} \approx \frac{1}{3}\theta_{12}^{\rm CKM} \approx \frac{1}{3}\theta_C \tag{4.44}$$

where the factor of 1/3 in eq. (4.44) originates from the GJ structure.

Using eqs. (4.42)–(4.44), the lepton mixing angle relations in eqs. (4.26)–(4.28) become

$$\theta_{13} \approx \frac{\theta_C}{3\sqrt{2}} \tag{4.45}$$

$$s_{12} \approx \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3}\theta_C \cos(\delta - \pi)\right) \tag{4.46}$$

$$s_{23} \approx \frac{1}{\sqrt{2}}\left(1 + \theta_{23}^{\rm CKM}\cos(\delta_3 - \delta_3')\right). \tag{4.47}$$

Eq. (4.45) gives a prediction for the reactor angle:

$$\theta_{13} \approx 3.06^\circ\,, \qquad \sin\theta_{13} \approx 0.052\,, \qquad \sin^2\theta_{13} \approx 2.7 \times 10^{-3}\,, \qquad \sin^2 2\theta_{13} \approx 1.1 \times 10^{-2}\,. \tag{4.48}$$

From eqs. (4.46), (4.47) the deviations from tri-bimaximal mixing may be expressed as:

$$\left|s_{12}^2 - \frac{1}{3}\right| \approx \left|\left(\frac{2}{9}\right)\theta_C \cos\delta\right| < 0.050 \tag{4.49}$$

$$\left|s_{23}^2 - \frac{1}{2}\right| \approx \left|\theta_{23}^{\rm CKM}\cos(\delta_3 - \delta_3')\right| < 0.042 \tag{4.50}$$

eq. (4.47) may also be expressed as

$$\theta_{23}^{\circ} \approx 45^{\circ} + \theta_{23}^{\rm CKM^{\circ}}\cos(\delta_3 - \delta_3') \tag{4.51}$$

which shows that the atmospheric angle is maximal $\theta_{23} = 45^{\circ}$ up to a correction no larger than $\theta_{23}^{\rm CKM} \approx 2.4^{\circ}$

$$\theta_{23} = 45^{\circ} \pm 2.4^{\circ}\,. \tag{4.52}$$

Eq. (4.46) leads to the tri-bimaximal complementarity relation in eq. (1.5):

$$\theta_{12}^{\circ} + \frac{\theta_C^{\circ}}{3\sqrt{2}}\cos(\delta - \pi) \approx 35.26^{\circ} \tag{4.53}$$

eq. (4.53) shows that the solar angle takes its tri-bimaximal value $\theta_{12} = 35.26^{\circ}$ up to a correction no larger than $\frac{1}{\sqrt{2}}\frac{\theta_{\rm C}}{3} \approx 3.06^{\circ}$,

$$\theta_{12} = 35.26^{\circ} \pm 3.06^{\circ}\,. \tag{4.54}$$

From eq. (4.45) and eq. (4.53) we find the sum rule:

$$\theta_{12}^{\circ} + \theta_{13}^{\circ}\cos(\delta - \pi) \approx 35.26^{\circ}\,. \tag{4.55}$$

Using eq. (4.46) we can predict the neutrino oscillation phase δ from a future accurate measurement of the solar angle θ_{12}:

$$\cos(\delta - \pi) \approx 13.3(1 - \sqrt{3}s_{12}) \tag{4.56}$$

or alternatively from eq. (4.55),

$$\cos(\delta - \pi) \approx \frac{35.26^{\circ} - \theta_{12}^{\circ}}{3.06^{\circ}}\,. \tag{4.57}$$

For example, from an accurate measurement of the solar angle of $\theta_{12} = 33^{\circ}$ we predict $\cos(\delta - \pi) = 0.74$ or $\delta = 222^{\circ}$.

The above results are subject to some theoretical corrections as follows. The renormalisation group running corrections in running from the GUT scale M_X to M_Z depend strongly on $\tan\beta$ but may be typically estimated for $\tan\beta \sim 40$ as:[4]

$$\theta_{12}(M_Z) - \theta_{12}(M_X) \sim 1^{\circ} \tag{4.58}$$

$$\theta_{13}(M_Z) - \theta_{13}(M_X) \sim -0.5^{\circ} \tag{4.59}$$

$$\theta_{23}(M_Z) - \theta_{23}(M_X) \sim 2^{\circ}\,. \tag{4.60}$$

In addition there is some theoretical error in the predictions of a similar magnitude due to the analytic formulae used, the small angle approximations, and the subleading operator corrections.

[4]These estimates have been made using the software packages REAP/MPT introduced in [26].

5. Conclusions

The poorly determined MNS parameters, when compared to the accuracy of the measured quark mixing angles, presents an opportunity to make testable predictions in the lepton sector by relating the lepton mixing parameters to the quark ones. In this paper we have shown how the neutrino mixing angles and oscillation phase can be predicted from tri-bimaximal neutrino mixing, corrected by charged lepton mixing angles which we relate to quark mixing angles via quark-lepton unification. The resulting predictions provide a probe of the high energy structure of unified theories.

We have shown how tri-bimaximal neutrino mixing can originate from the see-saw mechanism using sequential dominance. We gave the conditions for tri-bimaximal neutrino mixing to originate from sequential dominance, thereby providing a general and natural framework for this approach called constrained sequential dominance (CSD). We discussed a realisation of CSD based on SO(3) family symmetry and vacuum alignment, although there are other examples that are possible. We then constructed a realistic model of quark and lepton masses and mixings based on the SO(3) family symmetry and vacuum alignment, together with quark-lepton unification arising from a Pati-Salam gauge group. With the ingredients of tri-bimaximal complementarity in place, the MNS parameters were then predicted in terms of the CKM parameters. Although these predictions have been derived for the specific model presented, we would expect them to apply to a more general class of models based on real vacuum alignment which lead to tri-bimaximal complementarity.

The atmospheric angle is predicted to be approximately maximal $\theta_{23} = 45^{\circ}$, corrected by the quark mixing angle $\theta_{23}^{\mathrm{CKM}} \approx 2.4^{\circ}$, with the correction controlled by an undetermined phase in the quark sector. The solar angle is predicted by the tri-bimaximal complementarity relation: $\theta_{12} + \frac{1}{\sqrt{2}}\frac{\theta_{\mathrm{C}}}{3}\cos(\delta - \pi) \approx 35.26^{\circ}$, where θ_{C} is the Cabibbo angle and δ is the neutrino oscillation phase. The reactor angle is predicted to be $\theta_{13} \approx \frac{1}{\sqrt{2}}\frac{\theta_{\mathrm{C}}}{3} \approx 3.06^{\circ}$. The neutrino oscillation phase δ is predicted in terms of the solar angle to be $\cos(\delta - \pi) \approx (35.26^{\circ} - \theta_{12}^{\circ})/3.06^{\circ}$. These predictions can all be tested by future high precision neutrino oscillation experiments. Indeed the link between low energy neutrino parameters and quark-lepton unification provides a powerful theoretical motivation for performing high precision neutrino oscillation experiments. In particular the prediction of the neutrino oscillation phase in terms of the solar angle is a remarkable result, which motivates an accurate measurement of the solar angle. The theoretical prediction for the reactor angle could be tested with the next generation of superbeam or reactor experiments, and the prediction for the oscillation phase could be accurately tested at a Neutrino Factory.

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A. Conventions and mixing formalism

We shall use the conventions defined in [12]. The Dirac mass matrices of the charged leptons and neutrinos are given by $m_{LR}^{\mathrm{E}} = Y_{LR}^{\mathrm{E}} v_{\mathrm{d}}$, and $m_{LR}^{\nu} = Y_{LR}^{\nu} v_{\mathrm{u}}$ where $v_{\mathrm{d}} = \langle h_{\mathrm{d}}^{0} \rangle$ and

$v_{\rm u} = \langle h^0_{\rm u} \rangle$, and the lagrangian is of the form $\mathcal{L} = -\bar{\psi}_L Y_{LR} h \psi_R + H.c.$ The neutrino mass matrix m^ν_{LL} is given by the type-I see-saw mechanism as

$$m^\nu_{LL} = m^\nu_{LR} M^{-1}_{RR} m^{\nu\,T}_{LR} \,, \tag{A.1}$$

in terms of the Dirac neutrino mass matrix m^ν_{LR} and the heavy Majorana mass matrix M_{RR}. In this convention the effective Majorana masses are given by the lagrangian $\mathcal{L} = -\bar{\nu}_L m^\nu_{LL} \nu^c + H.c.$ The change from flavour basis to mass eigenbasis can be performed with the unitary diagonalization matrices $V_{E_{\rm L}}, V_{E_{\rm R}}$ and $V_{\nu_{\rm L}}$ by

$$V_{E_{\rm L}}\, m^{\rm E}_{LR}\, V^\dagger_{E_{\rm R}} = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}, \qquad V_{\nu_{\rm L}}\, m^\nu_{\rm LL}\, V^T_{\nu_{\rm L}} = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}. \tag{A.2}$$

The MNS matrix is then given by

$$U_{\rm MNS} = V_{e_{\rm L}} V^\dagger_{\nu_{\rm L}} \,. \tag{A.3}$$

We use the parameterization $U_{\rm MNS} = U_{23} U_{13} U_{12}$ with U_{23}, U_{13}, U_{12} being defined as

$$U_{12} = \begin{pmatrix} c_{12} & s_{12} e^{-i\delta_{12}} & 0 \\ -s_{12} e^{i\delta_{12}} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad U_{13} = \begin{pmatrix} c_{13} & 0 & s_{13} e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13} e^{i\delta_{13}} & 0 & c_{13} \end{pmatrix},$$
$$U_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} e^{-i\delta_{23}} \\ 0 & -s_{23} e^{i\delta_{23}} & c_{23} \end{pmatrix} \tag{A.4}$$

where s_{ij} and c_{ij} stand for $\sin(\theta_{ij})$ and $\cos(\theta_{ij})$, respectively. δ, the Dirac CP phase relevant for neutrino oscillations, is given by $\delta = \delta_{13} - \delta_{23} - \delta_{12}$.

The MNS matrix is thus constructed as a product of a unitary matrix from the charged lepton sector V^{E_L} and a unitary matrix from the neutrino sector $V^{\nu_L \dagger}$. Each of these unitary matrices may be parametrised as:

$$V^\dagger = P_2 R_{23} R_{13} P_1 R_{12} P_3 \tag{A.5}$$

where R_{ij} are a sequence of real rotations corresponding to the Euler angles θ_{ij}, and P_i are diagonal phase matrices. The Euler matrices are given by

$$R_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \tag{A.6}$$

$$R_{13} = \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix} \tag{A.7}$$

$$R_{12} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{A.8}$$

where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$. The phase matrices are given by

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\chi} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{A.9}$$

$$P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\phi_2} & 0 \\ 0 & 0 & e^{i\phi_3} \end{pmatrix} \tag{A.10}$$

$$P_3 = \begin{pmatrix} e^{i\omega_1} & 0 & 0 \\ 0 & e^{i\omega_2} & 0 \\ 0 & 0 & e^{i\omega_3} \end{pmatrix}. \tag{A.11}$$

Thus we write

$$V^{\nu_L\dagger} = P_2^{\nu_L} R_{23}^{\nu_L} R_{13}^{\nu_L} P_1^{\nu_L} R_{12}^{\nu_L} P_3^{\nu_L} \tag{A.12}$$

$$V^{E_L\dagger} = P_2^{E_L} R_{23}^{E_L} R_{13}^{E_L} P_1^{E_L} R_{12}^{E_L} P_3^{E_L} \tag{A.13}$$

in terms of independent angles and phases for the left-handed neutrino and charged lepton sectors distinguished by the superscripts ν_L and E_L.

The MNS matrix can be expanded in terms of neutrino and charged lepton mixing angles and phases to leading order in the charged lepton mixing angles which are assumed small:[5]

$$s_{23}e^{-i\delta_{23}} \approx s_{23}^{\nu_L} e^{-i\delta_{23}^{\nu_L}} - \theta_{23}^{E_L} c_{23}^{\nu_L} e^{-i\delta_{23}^{E_L}} \tag{A.14}$$

$$\theta_{13}e^{-i\delta_{13}} \approx \theta_{13}^{\nu_L} e^{-i\delta_{13}^{\nu_L}} - \theta_{13}^{E_L} c_{23}^{\nu_L} e^{-i\delta_{13}^{E_L}} - \theta_{12}^{E_L} s_{23}^{\nu_L} e^{i(-\delta_{23}^{\nu_L} - \delta_{12}^{E_L})} \tag{A.15}$$

$$s_{12}e^{-i\delta_{12}} \approx s_{12}^{\nu_L} e^{-i\delta_{12}^{\nu_L}} + \theta_{13}^{E_L} c_{12}^{\nu_L} s_{23}^{\nu_L} e^{i(\delta_{23}^{\nu_L} - \delta_{13}^{E_L})} - \theta_{12}^{E_L} c_{23}^{\nu_L} c_{12}^{\nu_L} e^{-i\delta_{12}^{E_L}} \tag{A.16}$$

where

$$\delta_{12}^{\nu_L} = \omega_1^{\nu_L} - \omega_2^{\nu_L} \tag{A.17}$$

$$\delta_{13}^{\nu_L} = \omega_1^{\nu_L} - \omega_3^{\nu_L} \tag{A.18}$$

$$\delta_{23}^{\nu_L} = \chi^{\nu_L} + \omega_2^{\nu_L} - \omega_3^{\nu_L} \tag{A.19}$$

$$\delta_{23}^{E_L} = -\phi_2^{E_L} + \phi_3^{E_L} + \phi_2^{\nu_L} - \phi_3^{\nu_L} + \chi^{\nu_L} + \omega_2^{\nu_L} - \omega_3^{\nu_L} \tag{A.20}$$

$$\delta_{13}^{E_L} = \phi_3^{E_L} - \phi_3^{\nu_L} + \omega_1^{\nu_L} - \omega_3^{\nu_L} \tag{A.21}$$

$$\delta_{12}^{E_L} = \chi^{E_L} + \phi_2^{E_L} - \phi_2^{\nu_L} - \chi^{\nu_L} + \omega_1^{\nu_L} - \omega_2^{\nu_L}. \tag{A.22}$$

In the quark sector an analagous procedure is followed. The Dirac mass matrices of the quarks are given by $m_{LR}^{\mathrm{D}} = Y_{LR}^{\mathrm{D}} v_{\mathrm{d}}$, and $m_{LR}^{\mathrm{U}} = Y_{LR}^{\mathrm{U}} v_{\mathrm{u}}$. The change from flavour basis

[5] Note that the sign of the last term in eq. (A.15) is reversed compared to the results quoted in [12]. I am grateful to Stefan Antusch for correcting these results.

to mass eigenbasis can be performed with the unitary diagonalization matrices $V_{D_\mathrm{L}}, V_{D_\mathrm{R}}$ and $V_{U_\mathrm{L}}, V_{U_\mathrm{R}}$ by

$$V_{D_\mathrm{L}}\, m^{\mathrm{D}}_{LR}\, V^\dagger_{D_\mathrm{R}} = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}, \qquad V_{U_\mathrm{L}}\, m^{\mathrm{U}}_{LR}\, V^\dagger_{U_\mathrm{R}} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}, \tag{A.23}$$

The CKM matrix is then given by

$$U_{\mathrm{CKM}} = V_{U_\mathrm{L}} V^\dagger_{D_\mathrm{L}} \,. \tag{A.24}$$

We use the standard parameterization $U_{\mathrm{CKM}} = R^{\mathrm{CKM}}_{23} U^{\mathrm{CKM}}_{13} R^{\mathrm{CKM}}_{12}$ where we label the quark parameters as CKM to distinguish them from the (unlabelled) lepton mixing angles. If the CKM angles are given predominantly by the down mixing angles, then we may use the analagous results to those quoted above to obtain the corrections coming from the up sector. Thus for example the analagous relations to eq. (A.14)–(A.22) apply in the quark sector also, with the replacements $\nu \to D$ and $E \to U$. In the quark sector the phases $\omega_i^{D_L}$, $\omega_i^{U_L}$ are all undetermined and are used to remove phases from the MNS matrix. In particular $\omega_i^{D_L}$ may be used to set the phases $\delta_{12}^{CKM} = \delta_{23}^{CKM} = 0$, with a single CKM phase remaining, $\delta^{\mathrm{CKM}} = \delta^{\mathrm{CKM}}_{13}$.

B. Vacuum alignment

In order to achieve the desired vacuum alignment in this model , we shall introduce the following additional superpotential terms:

$$\begin{aligned}
W_{\mathrm{SB}} \sim\ & A(\phi_1^2 - \Lambda_1^2) + B(\phi_2^2 - \Lambda_2^2) + C(\phi_3^2 - \Lambda_3^2) + && \text{(B.1)}\\
& + D\phi_1.\phi_2 + E\phi_1.\phi_3 + F\phi_2.\phi_3 + && \text{(B.2)}\\
& + L\phi_{12}.\tilde{\phi}_{12} + M\phi_{23}.\tilde{\phi}_{23} + && \text{(B.3)}\\
& + N\phi_{123}.\phi_{12} + O\phi_{123}.\phi_{23} + && \text{(B.4)}\\
& + P((\phi_{12}.\phi_1)(\phi_{12}.\phi_2) - \Lambda_9^2) + Q((\tilde{\phi}_{12}.\phi_1)(\tilde{\phi}_{12}.\phi_2) - \Lambda_{10}^2) + && \text{(B.5)}\\
& + R((\phi_{23}.\phi_2)(\phi_{23}.\phi_3) - \Lambda_{11}^2) + S((\tilde{\phi}_{23}.\phi_2)(\tilde{\phi}_{23}.\phi_3) - \Lambda_{12}^2) + && \text{(B.6)}\\
& + T((\phi_{123}.\phi_1)(\phi_{123}.\phi_2)(\phi_{123}.\phi_3) - \Lambda_{13}^2) && \text{(B.7)}
\end{aligned}$$

where $A, \dots T$ are SO(3) and Pati-Salam singlet superfields and Λ_i are independent heavy mass scales which we regard as arising from the VEVs of some SO(3) singlet fields. Such VEVs could arise from some radiative symmetry breaking mechanism, for example [16]. Λ_i^2 can be taken to be real and positive by a suitable phase choice for the fields $A, \dots T$ [21]. Note that in such an SO(3) theory with real VEVs all the D-terms will be automatically zero, and the vacuum alignment is then achieved purely from the F-terms being minimised to zero, up to soft supersymmetry breaking perturbations. It is straightforward to deduce the required quantum numbers of these superfields under the symmetry group $R \times Z_4^2 \times Z_3^2 \times$ U(1) from the quantum number assignments of the flavons in table 1, and the requirement that the superpotential terms given above are allowed.

The potential consists of F-terms of the form $|F_X|^2$, together with positive soft mass squareds for the flavon fields. Since Λ_i^2 are real and positive this results in real VEVs as discussed in [19, 21], which greatly simplifies the analysis, and crucially restricts the number of undetermined phases in the analysis, ultimately leading to a prediction for the neutrino oscillation phase. The purpose of the terms in eq. (B.1), (B.2) is for the F-terms $|F_X|^2$, with $X = A, \ldots F$, to be minimised by real three orthogonal VEVs for $\phi_{1,2,3}$ of the form given in eq. (3.5). In particular the terms in eq. (B.1) drive the VEVs to be non-zero, and the the terms in eq. (B.2) together with the soft positive mass squareds then lead to real and orthogonal VEVs of the form given in eq. (3.5). The purpose of the remaining terms is to align the VEVs of the remaining fields relative to these basis vectors, in order to achieve the alignment for ϕ_{23}, ϕ_{123} as shown in eq. (3.5), as follows.

To achieve the alignment of ϕ_{23} requires two fields $\phi_{23}, \tilde{\phi}_{23}$. The purpose of the terms in eq. (B.6) is to drive non-zero VEVs for these two fields in the $(2, 3)$ directions, and taken together with the positive soft mass squared terms,[6] lead to a potential which is minimised when magnitudes of the VEVs along each of the directions is equal, for each of the two fields $\phi_{23}, \tilde{\phi}_{23}$ separately. This is because for each of these flavons the potential takes the form $V = m^2_{soft}(y^2 + z^2) + b^2(yz - M_2^2)^2$, where all parameters are real and positive and y, z are the components of the VEVs along the $2, 3$ directions respectively. Such a potential is minimised by component VEVs of equal magnitude $y^2 = z^2$. The term in eq. (B.3) proportional to M then ensures that the two VEVs are orthogonal to each other, and without loss of generality this results in VEVs of the form:

$$\tilde{\phi}_{23} \sim \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad \phi_{23} \sim \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \tag{B.8}$$

Following analagous arguments, the terms in eq. (B.5), together with positive soft mass terms and the term proportional to L in eq. (B.3) leads, without loss of generality, to VEVs of the form:

$$\tilde{\phi}_{12} \sim \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \phi_{12} \sim \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}. \tag{B.9}$$

The alignment of ϕ_{123} is achieved by the term in eq. (B.7) which drives the VEV, and ensures that all three components of the VEV are non-zero, and taken together the soft mass terms and F-terms in the potential which result from these terms imply that the component VEVs must have equal magnitude. The terms in eq. (B.4) then align these components to be orthogonal to both ϕ_{12} and ϕ_{23}, resulting in the alignment assumed in eq. (3.5):

$$\phi_{123} \sim \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{B.10}$$

[6] I am grateful to Graham Ross and Ivo de Medeiros-Varzielas for suggesting the use of soft mass terms to achieve this alignment. The use of soft mass terms for alignment is also discussed in [16].

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